CONSTRUCTION OF ROTATABLE DESIGNS FROM FACTORIAL DESIGNS

By M. N. Das

I.A.R.S., New Delhi

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1. Introduction

Rotatable designs were introduced by Box and Hunter (1954). For the construction of these designs they used geometrical configurations. They obtained several second order rotatable designs by using the properties of regular configurations. Afterwards, Gardiner and others (1959) obtained through the same technique some third order designs for two and three factors and one design in four factors. They observed that through their technique designs with larger number of factors may be possible, but they will require very large number of points. Bose and Draper (1959) obtained some second order designs in three factors by using a different technique.

In the present paper second and third order rotatable designs with up to 8 factors have been obtained from factorial designs. The number of points in these designs including the third order designs are reasonably small.

2. Rotatable Designs

Let there be k factors or variates each at s levels. If a design be formed with N of the s^k treatment combinations, it can be written in the following $(N \times k)$ matrix which we shall hereafter call the design matrix, D:

Variates

$$D = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & & x_k \end{array} \\ \left[\begin{array}{cccc} x_{11} & x_{21} & \cdots & x_{k1} \\ x_{12} & x_{22} & \cdots & x_{k2} \\ x_{13} & x_{23} & \cdots & x_{k3} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ x_{1N} & x_{2N} & \cdots & x_{kN} \end{array}\right] \end{array}$$

A variate x_i has been associated with the i-th variate such that the entries in the i-th column of D below the variate are its values. The treatment combinations will sometimes also be referred to as the points of the design.

A design of the above form will be a rotatable design of order d if a polynomial response surface of degree d of the response, y as obtained from the treatments on the variates x_i $(i = 1, 2, \cdots k)$ can be so fitted that the variance of the estimated response from any treatment is a function of the sum of squares of the levels of the factors in that treatment combination. In other words, the variance of the estimated response at any point is a function of the square of the distance of the point from some suitable origin so that the variance of all responses at points equidistant from the origin is the same. When the response surface is of the second degree, that is, $d = 2$, such constancy of variance will be possible if the design points are so selected that the following relations hold:

Relation A:

$$\Sigma x_i = 0, \quad \Sigma x_i x_j = 0, \quad \Sigma x_i x_j^2 = 0, \quad \Sigma x_i^3 = 0, \quad \Sigma x_i x_j^3 = 0,$$

$$\Sigma x_i x_j x_k^2 = 0 \quad \text{and} \quad \Sigma x_i x_j x_k x_l = 0. \qquad (i \neq j \neq k \neq l).$$

Relation B:

$$\left.\begin{array}{l} \text{(i) } \Sigma x_i^2 = \text{constant} = N\lambda_2 \\ \text{(ii) } \Sigma x_i^4 = \text{constant} = 3N\lambda_4 \end{array}\right\} \text{ for all } i\text{'s.}$$

Relation C:

$$\Sigma x_i^2 x_j^2 = \text{constant for all pairs of } i \text{ and } j \ (i \neq j).$$

Relation D:

$\Sigma x_i^4 = 3 \Sigma x_i^2 x_j^2$ for all i and j.

Relation E:

$$\frac{\lambda_4}{\lambda_2^2} > \frac{k}{k+2}.$$

In the case of the third order rotatable designs the following further relations need also be satisfied:

Relation A_1:

Each of the sum of powers or each of the products of powers of x_i's, in which at least one power is odd, is zero.

Relation B_1:

$\Sigma x_i^6 = \text{constant} = 15N\lambda_6$.

Relation C_1:

(i) $\Sigma x_i^2 x_j^4 = \text{constant}$.

(ii) $\Sigma x_i^2 x_j^2 x_k^2 = \text{constant}$ $(i \neq j \neq k)$.

Relation D_1:

$$\left.\begin{array}{l}\text{(i) } \Sigma x_i^6 = 5 \Sigma x_i^2 x_j^4 \\ \text{(ii) } \Sigma x_i^2 x_j^4 = 3 \Sigma x_i^2 x_j^2 x_k^2\end{array}\right\} \text{ for all } i, j \text{ and } k \ (i \neq j \neq k).$$

Relation E_1:

$$\frac{\lambda_6 \lambda_2}{\lambda_4^2} > \frac{k+2}{k+4}.$$

The problem of construction of rotatable designs consists not only in selecting a set of N treatment combinations out of s^k combinations, but also in spacing properly the levels of the factors with a suitable origin on a suitably chosen scale such that the relations specified are satisfied. In ordinary factorial experiments the problem of spacing the levels does not arise. Hence for such designs the relative magnitudes of the different levels remain unspecified.

3. A Modified Form of Factorial Treatments

A factorial experiment involving three factors each at two levels is usually denoted as 2^3 and the levels as 0 and 1. Instead we may denote the levels as $x(p)$ and $x(q)$ where $x(p)$ like $x(q)$ means that p

is to be associated with x according to certain rule of association. We may call x as magnitude and p and q the associates of the magnitude. If $p = 1$ and $q = -1$ and the rule of association is multiplication, we get the levels as x and $-x$ out of the magnitude x. From this angle we may denote this factorial design as $(1 \times 2)^3$ where 1 stands for the magnitude x and 2, the two associates and the product sign, the rule of association. In general we may say that the design $(m_1 \times m_2)^k$ have $m_1 \times m_2$ levels formed of m_1 magnitudes and m_2 associates. The $(m_1 \times m_2)^k$ treatments can be obtained by 'multiplying' the m_1^k magnitude combinations with m_2^k associate combinations. For example, when $m_1 = 2$ and $m_2 = 2$, if the magnitudes are α and β and the associates 1 and -1, the magnitude combinations are $\alpha\alpha$, $\alpha\beta$, $\beta\alpha$ and $\beta\beta$ and the associate combinations, 11, $1 - 1$, -11, $-1\ -1$.

The 16 treatments can now be obtained by multiplying each of the magnitude combinations with the associate combinations. The way of multiplication is illustrated below: When a magnitude combination say $\alpha\beta$ is multiplied with the 4 associate combinations we get $\alpha\beta$, $\alpha - \beta$, $-\alpha\beta$ and $-\alpha -\beta$. If we call the contents of any combination of k factors, whether of magnitudes or associates, as its elements, the multiplication of a magnitude combination with an associate combination generates a treatment which consists of k levels being the products of corresponding elements in the two combinations.

Designs with $m_2 = 1$, reduce to the ordinary factorial designs with m_1 levels. If some of the magnitudes as also the associates be zero, all the treatments obtained in this way will not be distinct. But for our purpose we shall use only distinct treatments unless otherwise mentioned.

4. Rotatable Designs as Fractional Replicates of the Modified Factorial Designs

For the purpose of construction of rotatable designs from the modified factorial designs the associates should always be numbers which are deviates from their mean so that their sum is zero. As will be evident later, for reducing the size of the design, the number of associates should be as small as possible, and hence 1 and -1 are the most suitable associates for this purpose.

The modified factorial design corresponding to any factorial design, s^k can be written as $(m_1 \times 2)^k$ where $m_1 = s/2$ or $(s + 1)\ 2$ according as s is even or odd. Such a design is always available for any s, provided 0 be taken as one of the magnitudes when s is odd.

Having given one magnitude combination and n associate combinations of the design $(m_1 \times m_2)^k$, we have seen that n treatments can be obtained from their multiplication. This method of obtaining treatments from a magnitude combination will be referred to as multiplication.

If any magnitude combination of the design $(m_1 \times 2)^k$ contains p non-zero magnitudes, the number of distinct treatments obtainable from it by multiplication with the 2^k associate combinations will be only 2^p and not 2^k.

If n combinations of the 2^k associate combination, with the associates as 1 and -1 be taken to form the design matrix D, and the factors be denoted as $A_1, A_2, \cdots A_k$ such that the variate x_i corresponds to A_i, then Σx_i will be zero if the main effect A_i is not confounded in D, as the values of x_i in it are $+1$ and -1 in equal numbers. Similarly $\Sigma x_i x_j$ will be zero if the interaction $A_i A_j$ is not confounded in the n combinations. If each of the values of the variate x_i in the design be raised to r-th power, Σx_i^r will remain equal to Σx_i, if r is odd and every entry in the column will be unity if r is even. Thus, if in the design, $\Sigma x_i = 0$, then Σx_i^r is also zero when r is odd. Again as $\Sigma x_i^{2r} x_j$ is equal to Σx_j, the former will vanish if $\Sigma x_j = 0$. In terms of interaction we may express this fact by making the convention that $A_i^r = 1$ or A_i according as r is even or odd and then equating any sum of products, say, $\Sigma x_i^2 x_j^3 x_m^5$ with the interaction $A_i^2 A_j^3 A_m^5$ which is equivalent to $A_j A_m$. Thus, if the interaction $A_j A_m$ be unconfounded in the treatments, $\Sigma x_i^2 x_j^3 x_m^5$ and all other sums of products corresponding to the interaction $A_j A_m$ will be zero. If in these n associate combinations none of the main effects and interactions with less than 5 factors be confounded, they will satisfy all relations A which are necessary for second order rotatable designs. A little thought shows that all these relations will be satisfied by the treatments which are obtained by the multiplication of any one or more magnitude combinations with the n associate combinations in which no main effect or interaction with less than 5 factors are confounded. If in these n associate combinations no interaction with less than 7 factors be confounded, relations A_1 also will be satisfied. A set of n associate combinations in which no interaction with less than 5 factors in the case of second order sign and 7 factors for third under designs is confounded, will be called an unaffected set of combinations. Any magnitude combination containing only one magnitude that is of the form $\alpha\alpha \cdots \alpha$ will be called a homogeneous set. Any homogeneous set multiplied with n unaffected associate combinations will produce treatments which will satisfy all relations A, A_1, B, B_1 and C, C_1.

Having given the number of magnitudes for a design, if a magnitude combination contains more than one magnitude, we may obtain other magnitude combinations out of it by cyclically changing over the magnitudes. Thus if there be 4 magnitudes α, β, γ, δ and a combination $\alpha\beta\gamma\delta$ be taken, we shall obtain out of $\alpha\beta\gamma\delta$ by cyclically changing over magnitudes three other combinations, *viz.*, $\beta\gamma\delta\alpha$, $\gamma\delta\alpha\beta$ and $\delta\alpha\beta\gamma$. The magnitude combination, with which we start, will be called the initial set and the process by means of which other sets are generated out of it, as detailed above, will be called rotation over the magnitudes.

Given any initial set we shall obtain out of it a group of treatments by rotation and multiplication which will satisfy all relations A, A_1; B, B_1 and partly C, C_1.

If there be only two factors, relations C and C_1 do not appear. In the case of three factors each with three magnitudes the initial set containing all of them generate treatments by rotation and multiplication which will satisfy all relations C but will produce one equation from relations in C_1.

Just like the homogeneous set $\alpha\alpha \cdots \alpha$, the set $\alpha\beta\beta \cdots \beta$ subjected to rotation and multiplication will generate treatments which will satisfy all relations A, A_1; B, B_1 and C, C_1. The set $\alpha 00 \cdots 0$ is a particular case of $\alpha\beta\beta \cdots \beta$ and hence this set also satisfies all these relations. The treatment combination $0, 0, \cdots 0$ called the central point satisfies all relations excepting E and E_1.

Relation D can be satisfied in some cases by proper choice of several sets as will be illustrated afterwards. In general, relations D, D_1, C, C_1 are utilised to obtain a set of equations involving the magnitudes as unknowns, a solution of which gives the proper spacing of the magnitudes so as to satisfy relations D, D_1, C and C_1.

If there be k factors and the treatments are generated as described earlier the number of equations obtainable from the treatment for satisfying relations C is $(k-2)/2$ when k is even or the number just less, if k is odd. The number of equations from relations C_1 is $k-2$. Thus, when the number of factors is more, larger number of magnitudes are necessary for getting designs excepting when sets like $\alpha\alpha \cdots \alpha$ and $\alpha\beta\beta \cdots \beta$ only are used. Though second order designs are obtainable by increasing the number of magnitudes and by satisfying relations C with their proper choice, in the case of third order designs increase in magnitude is no easy remedy, particularly when there are four or more factors. In such cases unless all relations C and C_1 are satisfied by proper choice of sets, it may not always be possible to get designs. One method of generating sets out of any given set so that the treatments obtained from all these sets by multiplication, satisfy all relations A, A_1, B, B_1, C and C_1, has been described below.

Let there be k factors and let us take a set in which there are $s(\leqslant k)$ distinct magnitudes. If $s = k$, the total number of ways in which these magnitudes can be allotted to the k factors is evidently $\angle k$. If we form a set out of each allotment, the treatments obtained from these sets through multiplication will satisfy all these relations. Again if $s < k$, that is, when there are fewer magnitudes than factors, the total number of ways of allotment of them to the factors is evidently

$$\frac{\angle k}{\angle r_1 \angle r_2 \cdots \angle r_s}$$

where $r_1 + r_2 + \cdots + r_s = k$ and r_i denotes the number of times the i-th magnitude occurs in the starting set. For example, if there be 5 factors and two magnitudes and the starting set is taken as $\alpha\alpha\beta\beta\beta$, the total number of allotments, and hence sets, is $\angle 5/(\angle 2 \angle 3)$, as here $r_1 = 2$ and $r_2 = 3$. It will be found that this procedure requires too many sets and hence so many points in the designs obtained through them. One remedy is to take 0 as one of the magnitudes and repeat it $(k-2)$ times together with one more magnitude repeated twice. In this case there will be $k(k-1)$ sets giving $2k(k-1)$ points in all. This procedure of obtaining sets will be called permutaion. As an example, when $k = 5$, permutation of the set $\alpha\alpha000$ gives the following 10 sets.

$\alpha\alpha000$	$0\alpha\alpha00$	$00\alpha\alpha0$	$000\alpha\alpha$.
$\alpha0\alpha00$	$0\alpha0\alpha0$	$00\alpha0\alpha$	
$\alpha00\alpha0$	$0\alpha00\alpha$		
$\alpha000\alpha$			

It is known that relations E and E_1 cannot be satisfied when a design is obtained out of one set only through the processes, as in this case all points will have the same distance from the origin. In such situations at least one central point $(0, 0, \cdots 0)$ has to be taken together with the others to satisfy E, but not E_1.

It will be evident from the previous considerations of generating the design points that for keeping the size of a design low, the starting sets should be homogeneous and those non-homogeneous should contain smaller number of non-zero magnitudes together with zero. In the next section we shall discuss some of the second order designs in detail while for other designs only the starting sets and the solutions for the magnitudes will be indicated. The second order designs have been presented according to number of levels as in ordinary factorial designs.

5. Designs with Three Levels

The factorial design used for the construction of rotatable designs with 3 levels is $(2 \times 2)^k$ where the magnitudes are 0 and α and the associates 1 and -1. The magnitude combinations for two factors are therefore $\alpha\alpha$, $\alpha 0$, 0α and 00 and the associate combinations, 1 1, 1 -1, -1 1 and -1 -1. By taking the starting set $\alpha\alpha$, the treatments generated by the multiplication of it with all the associate combinations cannot satisfy relations D. As there is only one magnitude α which has to be fixed to make $\Sigma x_i^2 = N$, relation D has to be satisfied by suitable choice of the magnitude combinations. It can be seen easily that the treatments obtained from the sets (i) $\alpha\alpha$ and (ii) $\alpha 0$, when the second set is repeated 4 times, by rotation and multiplication, satisfy all the relations A, B, C, D and E. If necessary the design can be augmented by adding some central points, 00. The value of α has to be obtained from $\Sigma x_i^2 = N$ where N stands for the total number of design points including the central points. This design will then have at least 20 points, as shown below:

$\alpha\alpha$	$\alpha 0$	$\alpha 0$	$\alpha 0$	$\alpha 0$
$\alpha - \alpha$	$-\alpha 0$	$-\alpha 0$	$-\alpha 0$	$-\alpha 0$
$-\alpha\alpha$	0α	0α	0α	0α
$-\alpha - \alpha$	$0 - \alpha$	$0 - \alpha$	$0 - \alpha$	$0 - \alpha$

In the general case of k factors a second order rotatable design can always be obtained by generating treatments through the two processes from the two sets (i) $\alpha\alpha \cdots \alpha$ and (ii) $\alpha 0 \cdots 0$ when the second set is repeated n times where n denotes, as stated earlier, the minimum number of associate combinations in which no interaction with less than 5 factors is confounded. The value of α is given by $\alpha = \sqrt{N/3n}$.

The number of points in the design will be $(2k + 1)\, n$ excluding the central points which may or may not be added. It will be seen that the number of points required for such designs is very large. For example, when $k = 3$, 56 points are required. Thus, when there is no compelling reasons for adopting designs with three levels, designs with larger number of levels may be adopted to bring down the size of the design.

When $k = 3$, there is another design obtainable from the sets (i) $\alpha\alpha 0$ and (ii) $\alpha 00$, the second set being repeated twice. This design is small in size as it requires only 24 points. The value of α in this design is given by $\sqrt{N/12}$.

6. Designs with 4 Levels

The factorial design for obtaining such rotatable designs is again $(2 \times 2)^k$ with the magnitude α and β and associates 1 and -1. The only possible type of starting set in this case is $\alpha\alpha \cdots \alpha\beta\beta \cdots \beta$ where α is repeated r_1 times and β, r_2 times. As the magnitudes α and β are to be chosen so as to satisfy the relations $\Sigma x_i^2 = N$ and $\Sigma x_i^4 = 3\,\Sigma x_i x_j^2$, all relations C must be satisfied by proper choice of sets. We know that the set $\alpha\beta\beta \cdots \beta$ gives treatments through the two processes of rotation and multiplication which satisfy all relations, C. Hence for constructing the design with k factors we may take the set $\alpha\beta\beta \cdots \beta$ and generate the treatments through the processes of rotation and multiplication. Relation D gives the equation:

$$n\{\alpha^4 + (k-1)\,\beta^4\} = 3n\{2\alpha^2\beta^2 + (k-2)\,\beta^4\}.$$

Putting

$$s = \frac{\alpha^2}{\beta^2},$$

the equation reduces to

$$s^2 + k - 1 = 6s + 3(k-2)$$

whence

$$s = 3 \pm \sqrt{4 + 2k}.$$

When $k = 2$, there will thus be two designs for two values of s, in other cases there will be only one design, as negative value of s cannot provide any real solution for α and β. There will be kn points in the design with at least one central point for satisfying relation E.

From relation B (1) the value of β comes as

$$\beta^2 = \frac{N}{(s + k - 1)\,n}$$

where n is the number of associate combinations taken and N, the total numbers of design points including the central points.

From consideration of geometrical configuration this series of designs was obtained by Box and Hunter (1954) also.

7. Designs with 5 Levels

The factorial design $(3 \times 2)^k$ where the magnitudes are 0, α, β and the associates, 1 and -1, will be used to get the design with 5 levels.

The sets (i) $(\alpha\alpha \cdots \alpha)$ and (ii) $\beta 0 \cdots 0$ produce designs for all values of k through the two operations of rotation and multiplication. The number of points in the design is $n + 2k$.

From relation D we get the equation:

$$n\alpha^4 \times 2\beta^4 = 3n\alpha^4$$

which gives $s = 1/\sqrt{n}$ where $s = \alpha^2/\beta^2$ and n is the number of associate combinations used for multiplication. In this case we need not add any central point to satisfy relation E excepting when $k = 2$ and 4 as in these cases the distance of all the design points from the centre becomes the same. The value of β can be obtained from

$$\beta^2 = \frac{N}{ns + 2}.$$

These designs are the central composite rotatable designs. One more series of designs like the central ocmposite designs is available when there are 5 magnitudes. This series is obtainable from the sets (i) $\alpha\alpha \cdots \alpha$ and (ii) $\beta\beta 00 \cdots 0$ when the second set is permuted to give $k(k-1)/2$ sets. The number of treatments in the design will be $n + 2k(k-1)$ where n is as before the number of associate combinations used for multiplication. Relation D gives the equation,

$$n\alpha^4 + 4(k-1)\beta^4 = 3n\alpha^4 + 12\beta^4$$

i.e.,

$$s^2 = \frac{\alpha^2}{\beta^2} = \frac{2(k-4)}{n}.$$

When $k = 4$, α is 0 and hence the design is available from the second set having 24 points together with one central point. The value of β can be obtained from

$$\beta^2 = \frac{N}{ns + 4(k-1)}.$$

In addition some other designs with 5 levels are also available. For example, a design can be obtained from the set $0\alpha\beta$. Relation D gives the equation, $4(\alpha^4 + \beta^4) = 12\alpha^2\beta^2$.

Taking

$$s = \frac{\alpha^2}{\beta^2},$$

it becomes

$$s^2 + 1 = 3s$$

and hence

$$s = \frac{3 \pm \sqrt{5}}{2}.$$

As there are two values of s, two designs are possible. The value of β is obtainable from $\beta^2 = N/4(s+1)$ where N includes at least one central point.

When $k = 4$, designs with one set in which all the magnitudes are present but one of them is repeated, cannot be obtained by rotation and multiplication as relations C cannot be satisfied with such sets. It appears the same is also true when $k > 4$.

8. Designs with 6 Levels

Let the magnitudes be α, β and γ and the associates, 1 and -1. When there are two factors, a design can be obtained from the sets $\alpha\beta$, $\beta\gamma$ and $\gamma\alpha$ by multiplying each one of them with the four associate combinations. This design will have 12 points. From relations D we get the equation

$$\alpha^4 + \beta^4 + \gamma^4 = 3(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2).$$

Putting

$$s = \frac{\alpha^2}{\gamma^2} \text{ and } t = \frac{\beta^2}{\gamma^2},$$

it reduces to

$$s^2 + t^2 + 1 = 3(st + s + t).$$

As there are two unknowns in one equation, one of them can be taken arbitrarily so that the solution for the others is real and positive. The value of γ can be obtained from

$$4\gamma^2(s + t + 1) = N.$$

In case of three factors a design with the initial set $\alpha\beta\gamma$ can be obtained by rotation and multiplication with all the 8 associate combinations. There will be 24 points in the design with at least one central point. The equation from relation D is the same as in the case of the two-factor design. Hence the same solution of s and t will give the design but the value of γ will be different, *viz.*, $8\gamma^2(s+t+1)=N$. This design was also obtained by Bose and Draper (1959). When $k=4$, a design obtained from the set $\alpha\beta\gamma\gamma$ will have one equation from relation C and one from D. As there are three unknowns and three equations in all it may be possible to get such a design. Equation from relation C comes out as

$$\gamma^4+\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2=2(\alpha^2\gamma^2+\beta^2\gamma^2)$$

i.e.,

$$(\gamma^2-\alpha^2)(\gamma^2-\beta^2)=0.$$

This shows that relation C cannot be satisfied when α, β and γ are different. Hence no design with $k=4$ and 6 magnitudes is possible through rotation and multiplication, though it may be possible by permutation from one set. It appears that in the general case of k factors no design with one set is possible likewise as relations C cannot be satisfied unless two of the magnitudes are equal.

9. Designs with 7 Levels

Let the magnitude be α, β, γ and δ and the associates 1 and -1. When the number of factors is less than 4, designs with 7 or more levels have no special advantage over those obtainable from smaller number of levels, rather they suffer from the defect of requiring too many points.

(*i*) *When* $k=4$:

We could not get any design with 6 levels when k is greater than three. With 7 levels if we take the set $0\alpha\gamma\beta$ a design can be obtained by developing the set through rotation and multiplication. There will be 32 points with at least one central point to satisfy relation E.

Relations C and D give the equations

(i) $\alpha^2\gamma^2+\beta^2\gamma^2=2\alpha^2\beta^2$.

(ii) $\alpha^4+\beta^4+\gamma^4=6\alpha^2\beta^2$.

Putting

$$s=\frac{\alpha^2}{\gamma^2} \text{ and } t=\frac{\beta^2}{\gamma^2},$$

they reduce to

$$s + t = 2st.$$

$$s^2 + t^2 + 1 = 6st.$$

Solving these equations we get:

$$s = 3\cdot137 \text{ and } t = \cdot595.$$

γ can be obtained from the relation $8\gamma^2(s + t + 1) = N$.

(*ii*) *When* $k = 5$:

We may choose a set in which all the magnitudes are presented but one of them is repeated. By repeating 0 we shall evidently obtain a design with a smaller number of points. Thus, a design can be obtained from the set $00\alpha\gamma\beta$ by adding at least one central point. The number of points excluding the central point is 40. The equations from relations C and D are

(i) $\alpha^2\gamma^2 + \beta^2\gamma^2 = \alpha^2\beta^2$

and

(ii) $\alpha^4 + \beta^4 + \gamma^4 = 3\alpha^2\beta^2$.

Putting

$$s = \frac{\alpha^2}{\gamma^2} \text{ and } t = \frac{\beta^2}{\gamma^2},$$

we get

$$s + t = st$$

and

$$s^2 + t^2 + 1 = 3st.$$

Solving these equations we get

$$s = 3\cdot3706 \text{ and } t = 1\cdot4206.$$

The value of γ can be obtained from

$$8\gamma^2(s + t + 1) = N.$$

When there are six factors or more relations C will give 2 or more equations. Unless it is possible to satisfy all the relations C excepting one so as to get only one equation out of it, there will be more equations than unknowns. As such no design in general with 7 level seems possible for six or more number of factors through rotation and multiplication.

10. Designs with 6, 7 and 8 Factors

(i) $k = 6$.—A design for 6 factors can be obtained by taking the set $00\alpha\beta0\delta$ and developing it through rotation and multiplication. Relations C and D give the following equations:

(i) $\alpha^2\beta^2 = \beta^2\delta^2 = 2\alpha^2\delta^2$.

(ii) $\alpha^4 + \beta^4 + \delta^4 = 6\alpha^2\delta^2$.

Putting

$$s = \frac{\alpha^2}{\delta^2}, \quad t = \frac{\beta^2}{\delta^2}$$

they become

$$st = t = 2s$$

and

$$s^2 + t^2 + 1 = 6s.$$

The solutions are $s = 1$, $t = 2$.

This design has 48 points and at least one central point is necessary.

(ii) $k = 7$.—A design for 7 factors can be obtained by taking the set $000\alpha\beta0\delta$ and developing it through rotation and multiplication. Relations C and D give the following equations:

(i) $\alpha^2\beta^2 = \beta^2\delta^2 = \alpha^2\delta^2$.

(ii) $\alpha^4 + \beta^4 + \delta^4 = 3\alpha^2\delta^2$.

Putting

$$s = \frac{\alpha^2}{\delta^2}, \quad t = \frac{\beta^2}{\delta^2}$$

they beome

$$st = t = s$$

and

$$s^2 + t^2 + 1 = 3s.$$

The solutions are $s = 1$, $t = 1$.

This design has 56 points and at least one central point is necessary.

(iii) $k = 8$.—For a design in 8 factors take the set $000\alpha\beta\gamma 0\omega$ and develop it through rotation and multiplication. Relations C and D give the following equations:

(i) $\alpha^2\beta^2 + \beta^2\gamma^2 = \alpha^2\gamma^2 + \gamma^2\omega^2 = \beta^2\omega^2 = 2\alpha^2\omega^2$.

(ii) $\alpha^4 + \beta^4 + \gamma^4 + \omega^4 = 6\alpha^2\omega^2$.

Putting

$$\frac{\alpha^2}{\omega^2} = s, \quad \frac{\beta^2}{\omega^2} = t, \quad \frac{\gamma^2}{\omega^2} = u$$

they become

$$st + tu = su + u = t = 2s$$

and

$$s^2 + t^2 + u^2 + 1 = 6s.$$

As there are more independent equations than unknowns, we take one more set $(2\sqrt{p\omega})0000000$ involving another unknown p, together with the previous set. The relations C and D now give the equations:

$$st + tu = su + u = t = 2s$$

and

$$s^2 + t^2 + u^2 + 1 + 2p^2 = 6s.$$

The solutions are

$$s = (\sqrt{2} - 1)$$
$$t = 2(\sqrt{2} - 1)$$
$$u = \sqrt{2}(\sqrt{2} - 1)$$
$$p = 0{\cdot}332.$$

This design has 144 points and no central point is necessary.

11. Third Order Rotatable Designs in Two Factors

Gardiner *et al.* (1959) have constructed a series of 2 factor designs by taking equidistant points on two concentric circles. We could get a design with three magnitudes which do not come out as any particular case of their series. This design has 16 points and are obtainable from the three sets (1) $\alpha\beta$, (2) $\alpha\alpha$ and (3) $\gamma 0$ by developing them through rotation and multiplication. Relations D and D_1 give the equations:

$$8\alpha^4 + 4\beta^4 + 2\gamma^4 = 24\alpha^2\beta^2 + 12\alpha^4$$

and

$$8\alpha^6 + 4\beta^6 + 2\gamma^6 = 20(\alpha^2\beta^4 + \alpha^4\beta^2) + 20\alpha^6.$$

Putting

$$s = \frac{\alpha^2}{\beta^2} \quad \text{and} \quad t = \frac{\gamma^2}{\beta^2},$$

they become

$$2t^2 + 4 = 24s + 4s^2$$

and

$$2t^3 + 4 = 20(s^2 + s) + 12s^3,$$

i.e.,

$$t^2 = 2s^2 + 12s - 2$$

and

$$t^3 = 6s^3 + 10(s^2 + s) - 2.$$

Solving these equations:

$$s = \cdot 25538$$

and

$$t = 1\cdot 09320.$$

The value of β can be obtained from

$$\beta^2(8s + 2t + 4) = N.$$

As the points are not all equidistant from the centre, conditions E and E_1 are satisfied.

(ii) Designs with four magnitudes can be obtained from the sets (i) $\alpha\beta$ and (ii) $\gamma\delta$. The number of points in the design will be again 16.

Relations C and C_1 do not appear when $k = 2$.

Relations D and D_1 give the equations

$$4(\alpha^4 + \beta^4 + \gamma^4 + \delta^4) = 24(\alpha^2\beta^2 + \gamma^2\delta^2),$$

$$4(\alpha^6 + \beta^6 + \gamma^6 + \delta^6) = 20(\alpha^2\beta^4 + \alpha^4\beta^2 + \gamma^2\delta^4 + \gamma^4\delta^2).$$

Putting

$$s = \frac{\alpha^2}{\delta^2}, \quad t = \frac{\beta^2}{\delta^2} \quad \text{and} \quad u = \frac{\gamma^2}{\delta^2}$$

we get

$$s^2 + t^2 + u^2 + 1 = 6(st + u)$$

$$s^3 + t^3 + u^3 + 1 = 5(st^2 + s^2t + u^2 + u)$$

As there are two equations and three unknowns, one of them can be chosen arbitrarily.

By putting $t/s = u$, we find the equations become:

$$(s^2 + 1)(u^2 - 6u + 1) = 0$$

and

$$(s^3 + 1)(u + 1)(u^2 - 6u + 1) = 0.$$

Thus, if u be so chosen that

$$u^2 - 6u + 1 = 0,$$

i.e.,

$$u = 3 \pm \sqrt{8}$$

the relations are satisfied whatever s may be. Thus a series of design is available from the sets:

(i) $(\sqrt{s}\delta)$, $(\sqrt{su}\delta)$

(ii) $(\sqrt{u}\delta)$ (δ)

where $u = 3 \pm \sqrt{8}$ and s is arbitrary. δ can be fixed from the relation $4\delta^2(s+1)(u+1) = N$. If $s = 1$, relations E and E_1 will not be satisfied while for all other values of s, they will be satisfied.

12. Third Order Rotatable Designs for Three and More Factors

It has been seen that with the help of the sets (i) $\alpha\alpha \cdots \alpha$ and (ii) $\beta 00 \cdots 0$ second order central composite rotatable designs can be obtained. But no third order designs can be obtained with them as relations D_1 cannot be satisfied. But if one more set, *viz.*, $\gamma\gamma 00 \cdots 0$ permuted to give $k(k-1)/2$, sets of four points each be taken together with the above, designs are available when one more set out of the above three, which may have different magnitudes, is also taken.

The equations in the general case of k factors obtainable from the above three sets are:

$$m\alpha^4 + 2\beta^4 + 4(k-1)\gamma^4 = 3m\alpha^4 + 12\gamma^4$$

$$4\gamma^6 + m\alpha^6 = 3m\alpha^6$$

$$m\alpha^6 + 2\beta^6 + 4(k-1)\gamma^6 = 5m\alpha^6 + 20\gamma^6.$$

where m denotes the number of associate combinations in which no interactions with less than 7 factors are confounded.

Putting

$$\frac{\alpha^2}{\gamma^2} = s, \quad \frac{\beta^2}{\gamma^2} = t,$$

these become

$$t^2 = ms^2 + 2(4 - k)$$

$$s^3 = \frac{2}{m}$$

$$t^3 = 2ms^3 + 2(6 - k) = 4 + 2(6 - k).$$

As there are three equations and two unknowns, one more magnitude is to be taken to obtain their solution. There will thus be the following three cases according to the nature of the added set:

Case 1.—If a set $\delta 00 \cdots 0$, *i.e.*, of the form $\beta 00 \cdots 0$ be taken and $\delta^2/\gamma^2 = u$, the equation for this design based on the four sets will be

$$t^2 + u^2 = ms^2 + 2(4 - k)$$

$$s^3 = \frac{2}{m}$$

$$t^3 + u^3 = 4 + 2(6 - k).$$

Case 2.—If again the added set be $\omega\omega \cdots \omega$ being of the form $\alpha\alpha \cdots \alpha$ and $(\omega^2/\gamma^2) = v$, the equations will be:

$$m(s^2 + v^2) = t^2 - 2(4 - k)$$

$$s^3 + v^3 = \frac{2}{m}$$

$$t^3 = 4 + 2(6 - k).$$

Case 3.—Lastly if the added set be $xx00 \cdots 0$ which is of the form $\gamma\gamma 0 \cdots 0$, the equations become when $x^2/\gamma^2 = p$.

$$t^2 = ms^2 + 2(4 - k)(1 + p^2)$$

$$s^3 = \frac{2(1 + p^3)}{m}$$

$$t^3 = \{4 + 2(6 - k)\}(1 + p^3).$$

It will be seen that in each set of the above equations one of the unknowns, *viz.*, either s or t is automatically known. The equations in the first two sets are of the form:

$$s^2 + v^2 = A$$

and

$$s^3 + v^3 = B.$$

If the relation $A^3/2 \leq B^2 \leq A^3$ holds and A and B are positive, there will be a positive real solution for s and v and one of the solutions will lie between $\sqrt{A}$ and $\sqrt{A/2}$. Some of the designs obtainable for such sets for different values of k up to 8 have been presented below:

(i) *When* $k = 3$:

The following sets (i) $\alpha\alpha\alpha$, (ii) $\beta 00$ (iii) $\gamma\gamma 0$ and (iv) $\delta 00$ give a design which belongs to case 1. This design has 32 points and no central points are necessary.

The equations for the design are:

$$t^2 + u^2 = 8s^2 + 2 = 5 \cdot 174797$$

$$t^3 + u^3 = 10$$

$$s^3 = \tfrac{1}{4}, \quad \textit{i.e.,} \quad s = \cdot 62996.$$

The solutions for t and u are:

$$t = 2 \cdot 1090$$

$$u = \cdot 8526.$$

(ii) *When* $k = 4$:

The following sets (i) $\alpha\alpha\alpha\alpha$, (ii) $\beta 000$, (iii) $\gamma\gamma 00$ and (iv) $xxC0$ give a design which evidently belong to case 3. The equations for these sets are:

$$t^2 = 16s^2, \quad \textit{i.e.,} \quad t = 4s.$$

$$s^3 = \frac{1 + p^3}{8}$$

$$t^3 = 8\ (1 + p^3).$$

As the third equation follows from the first two, whatever p may be, there will be a design with these sets for each value of p. The values of t and s can be obtained as soon as p is fixed. When $p \neq 0$, there will be 72 points. If $p = 0$, there will be 48 points, but in this design all the points will be equidistant from the centre and hence relation E_1 cannot be satisfied even by adding central point.

(iii) *When* $k = 5$:

The following sets (i) $\alpha\alpha\alpha\alpha\alpha$, (ii) $\beta 0000$, (iii) $\gamma\gamma 000$ and (iv) $\omega\omega\omega\omega\omega$ will give a design in 114 points. This design belongs to case 2 and the equations are:

$$s^2 + v^2 = \frac{t^2 + 2}{32}$$

$$s^3 + v^3 = \frac{1}{16}$$

$$t^3 = 6, \quad \textit{i.e.,} \quad t = 1 \cdot 81712.$$

The solutions are $s = \cdot 3948$ and $v = \cdot 0991$.

(iv) *When* $k = 6$:

From the sets (i) $\alpha\alpha\alpha\alpha\alpha\alpha$, (ii) $\beta 00000$, (iii) $\gamma\gamma 0000$, but without any fourth set, a design can be obtained for which the equations are:

$$t^2 = 64s^2 - 4,$$

$$s^3 = \frac{1}{32}$$

$$t^3 = 4.$$

It is found that the values of s and t as obtained from the last two equations almost satisfies the first equation. Hence this design in 136 points is very near by a third order rotatable design.

If, however, one more set $\omega\omega\omega\omega\omega\omega$ be added, the design will have 200 points. It belongs to case 2 and the equations are

$$s^2 + v^2 = \frac{t^2 + 4}{64} = \cdot 10187$$

$$s^3 + v^3 = \frac{1}{32} \cdot$$

$$t^3 = 4.$$

Solving these equations:

$$t = 1 \cdot 58740$$

$$s = \cdot 31446$$

$$v = \cdot 05466.$$

(v) *When* $k = 7$:

The sets for a design are

(i) $\alpha\alpha\alpha\alpha\alpha\alpha\alpha$

(ii) $\beta 000000$

(iii) $\gamma\gamma 00000$

and

(iv) $\omega\omega\omega\omega\omega\omega\omega$.

In this design $m = 64$ and not 128. Hence there will be 226 points. This design belongs to case 2 and the equations are:

$$s^2 + v^2 = \frac{t^2 + 6}{64} = \cdot 11855$$

$$s^3 + v^3 = \frac{1}{32}$$

$$t^3 = 2.$$

Solutions to these equations are

$$t = 1 \cdot 2599$$

$$s = \cdot 2949$$

$$v = \cdot 1777$$

(vi) *When* $k = 8$:

In this case if we take the sets:

(i) $\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha$

and

(ii) $\gamma\gamma 000000$

the equations come out by taking $m = 128$,

$$s^2 = \frac{8}{m} = \frac{1}{16}$$

$$s^3 = \frac{2}{m} = \frac{1}{64}.$$

It is found that $s = \frac{1}{4}$ is a solution and hence a rotatable arrangement is possible with these sets. With this solution for s, we find $8\alpha^2 = 2\gamma^2$ and hence all the points are equidistant from the centre. This indicates that relation E_1 cannot be satisfied with these points even by adding central points. It appears through the present technique it is not possible to get a design, by adding any further sets belonging to any of the four types tried.

13. Third Order Sequential Rotatable Designs

The third order designs presented earlier in the paper cannot be fitted into a sequential programming of experimentation as they are, Gardiner *et al.* (1959) have shown that a third order rotatable design will be sequential if the treatments constituting the design can be divided into two groups to form the contents of two blocks such that the treatments in each group form a second order rotatable design with some central points, if necessary. After the treatments have been divided to form the two blocks, let Σ_1 and Σ_2 stand for the sum over the treatments in the first and second blocks respectively. Now, in addition to the relation which are to be satisfied by the third order rotatable designs one more relation, *viz.*, $\Sigma_1 x_i^4 = 3\Sigma_1 x_i^2 x_j^2$ must be satisfied for the design to be sequential, as this will ensure that each block is a second order rotatable design. We shall call this relation F.

A further condition for sequential designs is that

$$\frac{\Sigma_1 x_i^2}{\Sigma_2 x_i^2} = \frac{n_1 + n_{10}}{n_2 + n_{20}}$$

where n_1 and n_2 are the numbers of treatments exlcuding the central points in the two blocks respectively and n_{10} and n_{20} denote the central points which may be added to the two blocks.

As $\Sigma_1 x_i^2$ and $\Sigma_2 x_i^2$ are functions of the magnitudes which are known while obtaining the design, the above relation can always be satisfied by suitably choosing n_{10} and n_{20}.

A number of such designs for values of $k = 3$, 4, 5 and 6 have been presented below.

(i) *When* $k = 3$:

Block No.		*Block contents*	*No of treatments*
Block I	..	Set (i) $\gamma\gamma 0$	$n_1 = 18$
	..	Set (ii) $\delta 00$	
Block II	..	Set (i) $\alpha\alpha\alpha$	$n_2 = 22$
	..	Set (ii) $\beta 00$	
	..	Set (iii) $\omega\omega\omega$	

The equations from the relations D, D_1 and F are

$$8(\gamma^4 + \alpha^4 + \omega^4) + 2(\beta^4 + \delta^4) = 12\gamma^4 + 24(\alpha^4 + \omega^4)$$

$$8(\gamma^6 + \alpha^6 + \omega^6) + 2(\beta^6 + \gamma^6) = 20\gamma^6 + 40(\alpha^6 + \omega^6)$$

$$4\gamma^6 + 8(\alpha^6 + \omega^6) = 24(\gamma^6 + \omega^6)$$

$$8\gamma^4 + 2\delta^4 = 12\gamma^4.$$

Putting

$$s = \frac{\alpha^2}{\gamma^2}; \quad t = \frac{\beta^2}{\gamma^2}, \quad u = \frac{\omega^2}{\gamma^2} \quad \text{and} \quad v = \frac{\delta^2}{\gamma^2}$$

the equations become:

$$t^2 + v^2 = 2 + 8(s^2 + u^2)$$
$$t^3 + v^3 = 6 + 16(s^3 + u^3)$$
$$4(s^3 + v^3) = 1$$
$$v^2 = 2.$$

Solving these equations

$$s = 0 \cdot 6$$
$$t = 1 \cdot 92849$$
$$v = 1 \cdot 41421$$
$$u = 0 \cdot 32390.$$

As

$$\Sigma_1 x_i^2 = \gamma^2 (8 + 2v)$$
$$\Sigma_2 x_i^2 = \gamma^2 (8s + 8u + 2t)$$
$$n_1 = 18, \quad n_2 = 22$$

the relation

$$\frac{\Sigma_1 x_i^2}{\Sigma_2 x_i^2} = \frac{n_1 + n_{10}}{n_2 + n_{20}}$$

can be satisfied by suitably choosing n_{10} and n_{20} so that

$$\frac{8 + 2v}{8s + 8u + 2t} = \frac{18 + n_{10}}{22 + n_{20}}$$

where s, t, u and v are as obtained above. The value of γ is to be obtained from:

$$\gamma^2 (8 + 2v + 8s + 8u + 2t) = N$$

where

$$N = n_1 + n_2 + n_{10} + n_{20}.$$

(ii) *When* $k = 4$:

The series of designs presented in Section 12 under $k = 4$ is also sequential when the block contents are taken as below:

Block No.		*Block contents*	*No. of treatments*
Block I	..	Sets (i) $\alpha\alpha\alpha\alpha$	$n_1 = 24$
		(ii) $\beta 000$	
Blocks II	..	Sets (i) $\gamma\gamma 00$	$n_2 = 48$
		(ii) $xx00$	

The values of the magnitudes are the same as given for the design.

Here $n_1 = 24$, $n_2 = 48$ and n_{10} has to be taken greater than one.

(iii) *When* $k = 5$:

Block No.		*Block contents*	*No. of treatments*
Block I	..	Sets (i) $\alpha\alpha\alpha\alpha\alpha$	$n_1 = 52$
	..	(ii) $\beta 0000$	
	..	(iii) $\delta 0000$	
Block II	..	Sets (i) $\omega\omega\omega\omega\omega$	$n_2 = 72$
		(ii) $\gamma\gamma 000$	

The equations for the design are:

$$32(\alpha^4 + \omega^4) + 16\gamma^4 + 2(\beta^4 + \delta^4) = 3 \times 32(\alpha^4 + \omega^4) + 12\gamma^4$$

$$32(\alpha^6 + \omega^6) + 16\gamma^6 + 2(\beta^6 + \delta^6) = 5 \times 32(\alpha^6 + \omega^6) + 20\gamma^6$$

$$32(\alpha^6 + \omega^6) + 4\gamma^6 = 3 \times 32(\alpha^6 + \omega^6)$$

$$32\alpha^4 + 2(\beta^4 + \delta^4) = 3 \times 32\alpha^4.$$

Putting

$$s = \frac{\alpha^2}{\gamma^2}, \quad t = \frac{\beta^2}{\gamma^2}, \quad u = \frac{\omega^2}{\gamma^2}, \quad v = \frac{\delta^2}{\gamma^2}$$

the equations become

$$t^2 + v^2 = 32(s^2 + u^2) - 2$$

$$t^3 + v^3 = 64(s^3 + u^3) + 2$$

$$16(s^3 + u^3) = 1$$

$$t^2 + v^2 = 32s^2.$$

Solution to these equations are:

$$u = 0 \cdot 25$$

$$s = 0 \cdot 36056$$

$$t = v = 1 \cdot 44225$$

(iv) *When* $k = 6$:

Block No.		*Block contents*	*No. of treatments*
Block	I ..	Sets (i) $\omega\omega\omega\omega\omega\omega$	$n_1 = 124$
		(ii) $\gamma\gamma 0000$	
Block	II ..	Sets (i) $\alpha\alpha\alpha\alpha\alpha\alpha$	$n_2 = 136$
		(ii) $\beta 00000$	
		(iii) $\delta\delta 0000$	

The equations for the design are:

$$64(\alpha^4 + \omega^4) + 20(\delta^4 + \gamma^4) + 2\beta^4 = 3 \times 64(\alpha^4 + \omega^4) + 12(\delta^4 + \gamma^4)$$

$$64(\alpha^6 + \omega^6) + 20(\delta^6 + \gamma^6) + 2\beta^6 = 5 \times 64(\alpha^6 + \omega^6) + 20(\delta^6 + \gamma^6)$$

$$64(\alpha^6 + \omega^6) + 4(\delta^6 + \gamma^6) = 3 \times 64(\alpha^6 + \omega^6)$$

$$64\omega^4 + 20\gamma^4 = 3 \times 64\omega^4 + 12\gamma^4.$$

Putting

$$s = \frac{\alpha^2}{\gamma^2}, \quad t = \frac{\beta^2}{\gamma^2}, \quad u = \frac{\omega^2}{\gamma^2}, \quad v = \frac{\delta^2}{\gamma^2}$$

the equations become

$$t^2 = 64(s^2 + u^2) - 4(1 + v^2)$$

$$t^3 = 128(s^3 + u^3)$$

$$32(s^3 + u^3) = 1 + v^3$$

$$16u^2 = 1.$$

Solving these equations

$$s = 0{\cdot}4660, \quad t = 2{\cdot}4636, \quad v = 1{\cdot}3990 \text{ and } u = 0{\cdot}25.$$

Summary

Usually the rotatable designs are obtained from regular geometrical configuration. This makes it difficult to construct such designs with larger number of factors. In the present paper a modified form of factorial designs has been defined and they have been utilised to obtain rotatable designs in a simple manner. Second and third order

rotatable designs for each of the number of factors from 2 to 7, have been constructed. Some third order designs which fit into sequential programming of experimentation have also been evolved. The number of points required for the designs are reasonably small. The method has the speciality that it makes easier and systematic the investigation for obtaining such designs. Through this technique a large number of new designs could be obtained for up to 8 factors.

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